Procyclical leverage and endogeneous Value-at-Risk constraint *

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Abstract

This paper develops a new DSGE framework where (i) bank equity need arises endo-
genously and is modelled as a junior liability, (ii) the composition of banks’ liability side
depends on limited liability that generates secured versus unsecured funding and (iii) the
heterogeneity in bank size and leverage relies on a Hotelling spacial model on the funding
side. The endogenous Value-at-Risk constraint is such that the bank can save on its expen-
sive equity but at the cost of a larger risk premium on unsecured debt. The model features
procyclical market discipline and bank default risk and thus replicates stylised facts about
bank leverage : procyclical leverage in the time dimension as well as heterogeneity in bank
leverage with bigger banks more levered. I show that (a) loan portfolio quality shocks are
amplified by market forces that affect the composition of banks balance sheet, and (b) a
change in tail expectations about future aggregate shocks can generate real effects today via
larger bank equity requirements to dampen banks’ fragility.

Keywords : DSGE; endogenous Value-at-Risk; bank default risk; market discipline; tail
risk; endogenous financial instability.

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1 Introduction

Banks manage their portfolio actively, both on the asset and liability side. This observation is not compatible with the Modigliani-Miller theorem of irrelevance of the funding structure. It is rather consistent with financial institutions targeting a certain level of capital ratio driven by procyclical market discipline (Duprey and Lé, 2014) and thus generating procyclical leverage (Adrian and Shin, 2010). As a result bank capital regulation needs to rely on a theory of endogeneous bank equity that is not the mere result of exogenously set capital constraints. It may be attractive to study the elasticity of bank lending to variations in bank capital regulation, but, as in most DSGE models\footnote{Among many others: Gerali et al., 2010; Meh and Moran, 2010.}, decreasing bank equity to zero would be in fact the optimal choice...

This paper tries to fill in this gap by introducing a regulation-free model where bank leverage choice would be endogenously determined by market forces and thus replicate stylised facts about bank leverage both in the time and cross-sectional dimension. It requires first to introduce a non-trivial bank balance sheet structure that would allow moving away from the Modigliani-Miller world. Then bank portfolio risk together with bank equity limited liability generates banking instability that is priced in unsecured or senior debt. As a result the bank faces the following trade-off: in order to maximise its profits and return on equity, it should issue less equity to benefit from the upside risk. But its riskiness will impact its ability to repay unsecured debt: less equity funding increases the probability of the bank defaulting on its other debt holders for a given shock. With rational expectations about the distributions of future shocks, debt holders should require a larger risk premium that decreases the profits per unit of equity. Thus it provides the bank with an incentive to issue more equity despite the associated cost of doing so that is included in the way of a tax shield on debt\footnote{See the corporate finance literature on the capital structure implications of the tax shield, eg. Kemsley and Nissim (2002).}. Henceforth, there exists an equilibrium bank leverage (total bank size over equity) that would be associated with the bank being resilient to shocks on its portfolio return up to a certain point while defaulting on debt for larger shocks. This corresponds to the concept of Value-at-Risk (VaR thereafter) widely used in the finance industry to measure the riskiness of an institution: the worst portfolio return, arising with some probability, that can be absorbed by the equity cushion without triggering bank default. As such bank default in my setting is defined in a probabilistic way, while other papers with bank default usually have a continuum of financial institutions with a time-varying fraction defaulting each period\footnote{Among others Clerc et al., 2014; ?; ?}. Conversely, previous general equilibrium models that used Value-at-Risk approaches never allowed for banks default and always assumed that bank capital would be determined by the maximal amount of risk possible (eg. Aoki and Sudo, 2012).

This endogeneous Value-at-Risk is inherently procyclical as a negative productivity shock shifts the expected distribution of bank portfolio shocks to the left and thus makes subsequent losses more likely. Larger current and expected non-performing loans push for more equity holding, thus increases average bank funding costs and thus lowers banks’ lending. In the meantime, larger banks are more subject to this procyclicality as they leveraged more in the first place by relying on a larger pool of deposits and thus a cheaper access to unsecured debt. The heterogeneity in bank size in my setting is not due to the regulation nor ad-hoc assumptions about their specialisation but by technological constraints (Hotelling spacial model with travelling costs to local bank agencies) that made some banks more widely available to capture deposits. Only in a second step are larger banks able to benefit from a cheaper access to outside finance as the fixed cost of entering the world markets are smaller per unit of asset.

This general equilibrium model of tail risk allows me to investigate two largely overlooked types of shocks: (realised) loan portfolio quality shocks, as well as shocks on the expected (not...
realised) financial fragility of banks. Shocks on portfolio risk result from cyclical aggregate shocks in the productive sector leading to unexpected variations in non-performing loans, while non-performing loans arise as a result of idiosyncratic productivity shocks. In addition I show that a shock on the expected resilience of the banking sector has a long lasting and delayed effect on the real economy, while a standard productivity shock has an immediate effect: shocks originating in the banking sector and affecting their resilience are potentially more dangerous than standard productivity shocks.

The present model provides a theory of bank capital subject only to market forces, with default risks partly internalised via expectations. The present approach assumes that agents form expectations not only on the mean future state but also on the whole distribution of shocks; it then allows to calibrate the steady-state expectations of tail risk by matching systemic risks measures. Furthermore, this setting gives the possibility to study in a second step the introduction of balance sheet regulation, namely its ability to efficiently internalise risks with respect to the market equilibrium, but also its potential distortionary impact when misdesigned. It can also be used to analyse the effects of off-balance sheet practices or bailout expectations that would distort the expectations of tail portfolio shocks and thus the lower perceived risk would allow the bank to increase its leverage, raising its actual default probability. These financial stability implications are analysed in a companion paper.

In terms of policy implications, results suggest that (i) prudential regulators should be worried about the perception of market participants as to the resilience of financial institutions as expectations of fatter tail risks can generate an endogeneous process of bank deleveraging reducing the loan supply to the economy; (ii) countercyclical capital regulation can make sense if market participants have excessively negative tail risk expectations in order to relax the leverage constraint of banks. However it is unclear that banks would react to a looser regulatory leverage ratio in bad times as leverage is inherently procyclical, making a regulatory leverage ratio non-binding in case of downturn.

The next section recalls the major stylised fact of post-2008 financial economics I investigate here, namely the evolution of bank leverage both in the time and cross-sectional dimension. Section 3 describes the benchmark model with the endogenous Value-at-Risk constraint, absent any regulation. Then section 4 shows the implication of bank leverage channel for productivity shocks and expectation shocks. Section 6 concludes.
2 Stylized facts

The present paper intends to include two stylised facts into a DSGE framework. First bank leverage is procyclical ([Adrian and Shin, 2010]) and then larger banks tend to be more leveraged. For instance the non-bank sector is shown to be acyclical in a recent theoretical paper by Adrian and Boyarchenko (2013).
Figure 1: Average risk weighted capital ratio for European banks across size quantiles.
3 Benchmark model

3.1 Production side

**Final production sector.** The final producer buys each intermediate good \( y_{i,t} \) at price \( p_{i,t} \) and combines them in a CES fashion (\( \epsilon > 1 \)) into a homogeneous consumption good. Its production function \( Y_t \) is thus given by:

\[
Y_t = \left( \sum_{i=0}^{\frac{\epsilon-1}{\epsilon}} y_{i,t} \right)^{\frac{1}{\epsilon}}
\]

The final good producer maximises:

\[
\max_{y_{i,t}} \left\{ P_t Y_t - \frac{1}{0} p_{i,t} y_{i,t} d_i \right\}
\]

which gives the following demand function for each intermediate good:

\[
P_t Y_t^{1/\epsilon} y_{i,t}^{-1/\epsilon} = p_{i,t}
\]

where \( P_t \) is normalized to 1.

**Intermediate production sector.** There is a continuum of ex-ante identical firms that face an idiosyncratic productivity shock from a uniform distribution \( a_{i,t} \in U \rightarrow [a; 1 + a] \), as well as an aggregate productivity parameter which is autoregressive with innovations drawn from a normal distribution \( N \rightarrow [0; \sigma_t] \) where the variance is itself an AR(1) process. For simplicity, the aggregate productivity can also be written using a standard normal shock \( \xi_t \in N \rightarrow [0; 1] \) whose innovations are scaled up or down by the variance \( \sigma_t \):

\[
\ln \Gamma_t = \rho \ln \Gamma_{t-1} + \sigma_t \cdot \xi_t
\]

The individual production function, which requires both labour \( l_i \) and capital goods \( k_{i,t} \) is given by:

\[
y_{i,t} = a_{i,t} \Gamma_t k_{i,t-1}^{\alpha} l_{i,t-1}^{1-\alpha}
\]

which allows to rewrite the aggregate production function 1 as:

\[
Y_t = \Gamma_t k_{i,t-1}^{\alpha} l_{i,t-1}^{1-\alpha} \left( \frac{\epsilon}{2\epsilon - 1} \right)^{1/\epsilon}
\]

The maximisation program of an individual intermediate producer corresponds to the revenue from selling the production to the final good producer as well as selling back the depleted capital at the end of the production period at price \( P^K \); the intermediate producer has to pay the labour cost as well as the cost of capital goods bought at price \( P^K \) and the cost of bank loan which allows to cover the start-up cost at the beginning of the period, i.e. the entrepreneur needs to buy capital goods before starting the production. Henceforth, the intermediate producer maximises its profit subject to the demand function 3 and the budget constraint at the beginning of the period:
\[
\begin{align*}
\max_{k_{i,t}, l_{i,t}} & \quad \mathbb{E} \left( \Pi_{i,t+1}^c \right) \\
\max_{k_{i,t}, l_{i,t}} & \quad \mathbb{E} \left\{ p_{i,t+1} y_{i,t+1} - w_{l_{i,t}} - k_{i,t} P^k_t + P^K_{t+1} (1 - \delta) k_{i,t} + B_{i,t} - B_{i,t} (1 + r_{B,t}) \right\} \\
\text{subject to:} & \\
p_{i,t} & = P Y^{1/\epsilon}_{i,t} - l_{i,t}^{-1/\epsilon} \\
B_{i,t} & = k_{i,t} P^k_t \\
y_{i,t} & = a_{i,t} \Gamma_{k_{i,t-1} l_{i,t-1}^{1-\alpha}}
\end{align*}
\]

Recognizing that the individual subscript can be dropped as entrepreneurs are ex-ante identical, before the idiosyncratic uncertainty is revealed, and plugging back the two constraints as well as the production function, the first order conditions write:

\[
1 + r_{B,t} = \frac{\mathbb{E}(p_{t+1})}{P^k_t} \frac{\epsilon - 1}{\epsilon - 1} a_{i,t}^{-1/\epsilon} + \frac{\mathbb{E}(P^K_{t+1})}{P^K_t} (1 - \delta) \\
w_t = \frac{\mathbb{E}(p_{t+1})}{P^k_t} \frac{\epsilon - 1}{\epsilon} (1 - \alpha) \frac{\mathbb{E}(y_{t+1})}{l_t}
\]

The expected price of an intermediate good that is needed in both FOC is given by replacing the total production \(Y\) and the firm specific production function \(y_{i,t}\) in 3. Then taking the expectation, one gets:

\[
p_{i,t} = \left( \frac{\epsilon}{2\epsilon - 1} \right)^{\frac{1}{\epsilon - 1}} a_{i,t}^{-1/\epsilon} \\
\mathbb{E}(p_{t+1}) = \left( \frac{\epsilon}{2\epsilon - 1} \right)^{\frac{1}{\epsilon - 1}} \frac{\epsilon}{\epsilon - 1}
\]

**Default of intermediate producer.** An individual firm defaults if the value of its production net of wages is not enough to repay the loan to the bank net of the residual value of the capital goods. Thus there exists an idiosyncratic productivity threshold \(a_{i,t}\), given a specific aggregate productivity level, below which a firm is not productive enough to repay its bank loan. That is to say the firm defaults when its end of period profit becomes negative, i.e. after it completed its production affected by the idiosyncratic productivity shock:

\[
p_{i,t} y_{i,t} - w_{l_{i,t-1}} l_{i,t-1} < B_{t-1} (1 + r_{B,t-1}) - P^K_t (1 - \delta) k_{t-1}
\]

Together with equations 10 and 5, the previous inequality is rewritten:

\[
a_{i,t} < \left[ \frac{B_{t-1} (1 + r_{B,t-1}) + w_{l_{t-1}} k_{t-1} - P^K_t (1 - \delta) k_{t-1}}{\left( \frac{\epsilon}{2\epsilon - 1} \right)^{\frac{1}{\epsilon - 1}} \Gamma_{k_{t-1} l_{t-1}^{1-\alpha}}} \right]^{\frac{1}{\epsilon - 1}} \equiv a_t
\]

Note that if an intermediate good producer receives a bad productivity shock so that it falls below the default threshold given a certain level of aggregate productivity, it will default on the whole bank loan and the firm will be liquidated. The bank will seize all the remaining revenues
of the firm, net of wages which cannot be defaulted upon for sake of simplicity, and net of a proportional destruction cost $1 - \gamma$ for removing the installed capital.

As a result, the aggregate profits of the intermediate production sector which is subject to default in a monopolistic competition setting, is given by individual profits of non-defaulting firms, which is a sort of limited liability for entrepreneurs. As entrepreneurs are ex-ante identical before the idiosyncratic shock is revealed, the only difference between entrepreneurs is the shock they face and whether they are above or below the default threshold:

$$\Pi^e_t = \int_{0}^{1} \Pi^e_{t,t} dt$$

$$= \left(\frac{\epsilon}{2\epsilon - 1}\right)^{\frac{\epsilon}{\epsilon - 1}} \Gamma(\alpha) 1^{1-\alpha} \left(1 - \frac{2\epsilon - 1}{2\epsilon}\right) - (w_{t-1}l_{t-1} + k_{t-1}(1 + r_{B,t-1}) - P^K_t (1 - \delta)k_{t-1} - 1 - a_t)$$

(14)

**Capital goods producers.** Capital goods producers are not formally needed for the purpose at hand, but are included in order to make the design of the production sector more coherent with the fact that some intermediate goods producers default each periods, which prevents us to include in a meaningful way capital accumulation for each intermediate producers. Instead of reallocating capital goods arbitrarily across surviving firms, I assume that each intermediate producer buys capital goods in a competitive market and has the possibility to re-sell non-depreciated capital provided it was not destroyed following a bankruptcy. So the role of the capital good producer is to refurbish remaining capital into new one, with a simple one-for-one technology, and to invest to create new additional capital goods out of consumption goods subject to some adjustment cost. The production function is thus:

$$k_t = (1 - a_t(1 - \gamma))(1 - \delta)k_{t-1} + I_t \left[ 1 - \frac{\kappa}{2} \left( \frac{I_t}{I_{t-1}} - 1 \right)^2 \right]$$

(15)

And capital good producers choose their level of new investment in a perfectly competitive market.

$$\max_{I_t} \mathbb{E}_{\infty} \sum_{t=0}^{\infty} Q_t \left( P^K_t k_t - P^K_t (1 - a_t(1 - \gamma))(1 - \delta)k_{t-1} - I_t \right)$$

where $Q_t$ is the household stochastic discount factor and the consumption good being the numéraire, it is normalised to one. The FOC which resembles a Tobin’s Q type of equation is then:

$$1 = P^K_t \left( 1 - \frac{\kappa}{2} \left( \frac{I_t}{I_{t-1}} - 1 \right)^2 - \kappa \left( \frac{I_t}{I_{t-1}} - 1 \right) \frac{I_t}{I_{t-1}} \right) + Q_{t+1}P^K_{t+1}\kappa \left( \frac{I_{t+1}}{I_t} - 1 \right) \left( \frac{I_{t+1}}{I_t} \right)^2$$

(16)

Equilibrium prices are obtained by imposing the following free entry/zero profits condition:

$$P^K_t k_t = P^K_t (1 - a_t(1 - \gamma))(1 - \delta)k_{t-1} + I_t$$

(17)
3.2 Households

\[
\max_{C_t, L_t, D_t, K_t} \sum_{t=0}^{\infty} \beta^t \left( \ln (C_t - hC_{t-1}) + \Psi \ln (1 - L_t) \right) \tag{18}
\]

The representative household consumes \( C_t \), provides labour \( L_t \) to the firms, and saves by using either unsecured deposits \( D_t \) or equity bank capital \( K_t \). Overall, the household obtains a banks’ return \( R^b_t \) and the lump sum transfers from the tax shield on bank’s debt (non Modigliani-Miller wedge), so the budget constraint writes:

\[
(1 + \tau_t)C_t + D_t + K_t \leq w_{t-1}L_{t-1} + R^b_t + \Pi^e_t + T_t
\]

Here I keep a representative household even though I allow for the possibility to invest in two types of assets, namely unsecured debt and equity, which could possibly held by agents with different risk preferences. But by ensuring that my representative household is composed of both debt holders and equity investors, I can by-pass the difficulty of bank equity limited liability: it is here equivalent whether the debt holders or equity investors gain/lose due to limited liability as both types have a sort of intra-household solidarity. If this feature prevents me from looking at the wealth redistribution effect of the model, it is not trivial as the debt/equity choice for the bank which borrows funds from the households will be driven from the relative cost of the two assets and their probability of default.

For this setting to be relevant, I need to depart from the Modigliani-Miller (thereafter MM) theorem of the indifference of the bank’s balance sheet structure. To that extent, I consider that equity holders require a premium over the debt instrument; this can be understood as the counterpart of the inherent fluctuation of equity return while debt is repaid at a fixed rate (except in the few cases where the bank defaults), or this can result from a tax shield for debt which is typically deductible while equity is not, or one can simply consider that the household has to pay a tax on its capital holding which requires a larger gross return of bank equity. As a result, I set \( \mu^b_t = r^b_t + \chi \) where \( \chi \) represents the non-MM wedge over the return on unsecured debt.

Due to the limited liability constraint for bank equity, the budget constraint of households can be decomposed into three cases, where \( \Gamma_{VaR^{sec}} \) is the threshold value of the aggregate productivity shock below which the bank defaults (to be defined later):

1. if the bank faces a good macro shock, that is to say a large productivity of the firms it finances \( \Gamma^{VaR^{sec}} \) which reduces the delinquency rate of its loan book : the return of the investment of households into the bank is high and the bank can repay high dividends \( \tilde{\Pi}^b_t K_{t-1} = 1 + \tilde{\mu}^b_t > 1 + \mu_{t-1} \) where \( \mu_{t-1} = r_{d,t-1} + \chi \), that is to say the promised expected dividend stream can be rewritten as being equivalent to the cost of unsecured debt plus a risk premium \( \chi \).

\[
R^b_t = (1 + r_{d,t-1})D_{t-1} + \left( \frac{\tilde{\Pi}^b_t}{K_{t-1}} \right) K_{t-1}
\]

and the repayments of the bank to the household can be decomposed as follows, by distinguishing the expected equity return at the previous period from the non-expected equity return realized the following period once the uncertainty is resolved:

\[
R^b_t = (1 + r_{d,t-1})D_{t-1} + \left( 1 + \mu^b_{t-1} \right) K_{t-1} + \frac{\Pi^b_t}{K_{t-1}} K_{t-1}
\]

\[
\text{expected dividend} \quad \text{unexpected dividend}
\]
2. if the bank faces a bad macro shock (non-diversifiable), that is to say a smaller aggregate productivity of the firms it finances $\Gamma > \Gamma_{V a R_{sec}}$ which increases the delinquency rate of its loan book: then bank equity will absorb the negative shock such that not only the dividend can be reduced but the principal may not be repaid either, with $\frac{\bar{\Pi}_t^b}{\bar{K}_{t-1}} = 1 + \mu_t^b < 1 + \mu_{t-1}^b = 1 + r_{d,t-1} + \chi$

$$R_t^b = (1 + r_{d,t-1})D_{t-1} + \left(\frac{\bar{\Pi}_t^b}{\bar{K}_{t-1}}\right) K_{t-1}$$

and the repayments of the bank to the household can be decomposed as follows, again by distinguishing the expected equity return at the previous period from the non-expected capital loss where $\Pi_t^b < 0$:

$$R_t^b = (1 + r_{d,t-1})D_{t-1} + \left(1 + \mu_{t-1}^b\right) K_{t-1} + \frac{\Pi_t^b}{\bar{K}_{t-1}} K_{t-1}$$

3. if the bank faces an extreme macro shock, it is too large to be absorbed by its equity holders who benefit from limited liability. So the much smaller aggregate productivity $\Gamma < \Gamma_{V a R_{sec}}$ of the firms financed by the bank increases the delinquency rate of its loan book such that the bank is forced to default on a share of its unsecured deposits and goes bankrupt: partial default (of a share $0 < d_{s,t} < 1$) on unsecured debt:

$$R_t^b = (1 - d_{s,t})(1 + r_{d,t-1})D_{t-1}$$

and the repayments of the bank to the household can be decomposed as follows, where $\Pi_t^b = -(1 + \mu_t^b)K_{t-1} < 0$:

$$R_t^b = (1 + r_{d,t-1})D_{t-1} + \left(1 + \mu_{t-1}^b\right) K_{t-1} + \frac{\Pi_t^b}{\bar{K}_{t-1}} K_{t-1} - d_{s,t}(1 + r_{d,t-1})D_{t-1}$$

Henceforth, in order to encompass the three cases above-mentioned, for the representative household that combines unsecured depositors and equity holders, the budget constraint in expected terms, with the consumption tax $\tau$ to cover potential bank’s rescue, can be rewritten as follows:

$$(1 + \tau_t)C_t + D_t + K_t \leq w_{t-1}L_{t-1} + \left((1 + r_{d,t-1})D_{t-1} + (1 + \mu_{t-1}^b)K_{t-1}\right)(1 - d_t) + \Pi_t^b + \Pi_t + T_t(19)$$

This expression takes into account the fact that in expected terms, the bank will default with probability $d$, depending on its capitalisation, which will be detailed later, so that the distinction between the three cases above is endogenous. This specific setting allows me to keep the limited liability structure of bank capital while making it easy to solve with traditional linear techniques, at the cost of shutting down the possible effect of wealth reallocation across different types of households in case of partial bank default.

The first order conditions are given by:

$$1 = \lambda_t^b (1 + \tau_t)(C_t - hC_{t-1}) + \beta h C_t - hC_{t-1} \left(\frac{C_t - hC_{t-1}}{E_{C_t+1}-hC_t}\right)$$  \hspace{1cm} (20)

$$\Psi = w_t(1 - L_t)\beta \lambda_t^{h_{t+1}}$$  \hspace{1cm} (21)

$$\frac{\lambda_t^b}{\beta \lambda_t^{h_{t+1}}} = (1 + r_t^d)(1 - d_{t+1})$$  \hspace{1cm} (22)

$$\chi + r_t^d = \mu_t^b$$  \hspace{1cm} (23)
where $\lambda^h$ is the Lagrange multiplier associated with the budget constraint of the representative household.

Thus with the wedge $\chi > 0$ which generates an excessive cost of equity, the MM theorem of the indifference of a banks’ liability structure does not hold anymore, which will introduce a constraint on the amount of capital chosen by the bank to operate: by minimizing its expensive equity position, the bank will make a non-trivial leverage choice.

### 3.3 Banking sector

**Maximisation program.** The bank maximises its profits, that is to say the difference between the loans to the mass one of ex-ante identical entrepreneurs, where a fraction $a_{t+1}$ of entrepreneurs are expected to face a shock negative enough to trigger bankruptcy, while the bank finances itself by raising unsecured debt $D_t$ or bank equity $K_t$ and incur a capital adjustment cost $\phi$:

$$
\max_{B_t, D_t, K_t} \sum_{t=0}^{\infty} \beta^t \{ \Pi_t \} 
$$

$$
\sum_{t=0}^{\infty} \beta^t \left\{ \int_{a_{t+1}}^1 (1 + r_{B,t}) B_t d\bar{i} - (1 + r_{d,t}^f) D_t - (1 + \mu_{t}^b) K_t + bkset_{t+1} - \frac{\phi}{2} (K_t - K_{t-1})^2 \right\}
$$

In addition, the bank gets the residual after firms’ bankruptcy settlement $bkset$ in a lump sum fashion as it is in practice hard to predict and the settlement process is uncertain. It is given by the revenues from selling the production of the bankrupt firm net of wages repayment which cannot be defaulted upon, as well as the revenues from liquidating the capital used for production with a loss of $1 - \gamma$:

$$
bkset_t = \int_0^{a_{t+1}} \left( p_{i,t} \Gamma_t k_t a_{t,t}^{\alpha-1} a_{t,t} - w_{t,l} l_t + P_{t+1}^{K} (1 - \delta) \gamma k_t \right) d\bar{i}
$$

$$
= \Gamma_t k_t a_{t,t}^{\alpha-1} \left( \frac{\epsilon}{2\epsilon - 1} \frac{2\epsilon - 1}{2\epsilon} a_{t+1} - w_{t,l} a_{t+1} + P_{t+1}^{K} (1 - \delta) \gamma k_{t+1} \right)
$$

The bank optimises subject to its balance sheet constraint:

$$
\int_0^1 B_t d\bar{i} = B_t = D_t + K_t
$$

Also, the bank faces a leverage constraint which stems from a Value-at-Risk (VaR) constraint. The bank defaults if its revenue does not cover the repayment of the unsecured debt, without taking into account the revenues from foreclosed firms that accrues to the bank only at the end of the period (it takes time to settle a bankruptcy):

$$
(1 + r_{B,t})(1 - a_{VaR,t}) B_t \leq (1 + r_{d,t}^f) D_t
$$

Let’s note $1 + \Theta \equiv (1 + r_{B,t})(1 - a_{VaR,t})$ the total return of bank loans that the capitalisation of the bank can sustain. Here $a_{VaR,t}$ corresponds to the threshold share of bankrupt firms that
allows the bank to repay debt holders while capital holders are completely wiped out. In other words, this threshold can be understood as corresponding to the maximal default risk debt holders accept to hold: above $a_{VaR,t}$, the bank defaults on its debt holders. Together with the balance sheet identity, it can be rewritten in the form of a leverage constraint, that is to say in order for uninsured debt holders to be repaid if the share of bankrupt firms increases, the bank must reduce its leverage by increasing its use of equity. The leverage constraint, which boils down to a leverage target as in Duprey and Lé (2014) for a given probability of default, is given by:

\[
\frac{K_t}{D_t} \geq \frac{1 + r^d_t}{1 + \Theta} - 1
\]  

(27)

One can see that as the amount of core liabilities increases, the bank must hold more capital. The FOC for the choice of uninsured debt $D_t$ is obtained after removing $B$ using the balance sheet identity and $K$ using the leverage constraint. It features the equality between the marginal gain of having an additional unit of loan (thanks to both more capital and more deposits whose share is imposed by the endogenous leverage constraint) and the marginal cost of having an additional unit on the balance sheet with has to be financed both by additional uninsured debt and also additional capital which bears some costs of adjustment $\phi$. The FOC writes fully:

\[
\frac{(1 + r_{B,t})(1 - a_{t+1})}{1 + \Theta} = 1 + \frac{1 + \mu^t_k + \phi(K_t - K_{t-1}) - \beta \phi(K_{t+1} - K_t)}{1 + r_{D,t}} \left(1 + \frac{1 + r_{D,t}}{1 + \Theta} - 1\right)
\]  

(28)

**Exogenous VaR.** I first show the case of an exogenous VaR constraint. The threshold $a_{VaR,t}$ which corresponds to the maximal share of non-performing loans the bank can absorb without defaulting, associated to a Value-at-Risk $VaR$, is given as in equation 13 by:

\[
a_{VaR,t} = \frac{k_t (1 + r_{B,t}) + w_t l_t - P^t_{K_{t+1}}(1 - \delta)k_t}{\mathbb{E}(p_{t+1}) \Gamma_{VaR,t+1} k_t^{\alpha} l_t^{-\alpha}}
\]  

(29)

but here $\Gamma_{VaR,t+1}$ corresponds to the aggregate productivity in case the economy hits a tail shock, and is given by using the law of motion of aggregate productivity equation 4:

\[
\Gamma_{VaR,t+1} = \exp(\rho \ln \Gamma_t - VaR \cdot \sigma_{t+1})
\]  

(30)

where $VaR$ is the actual Value-at-Risk constraint which can be given exogenously if one recalls that $\epsilon \in \mathbb{N} \rightarrow [0; \sigma_t = 1]$; that is to say $-VaR \cdot \sigma_{t+1}$ corresponds to a threshold shock $a_{VaR,t}$ below which the bank defaults. The probability of the aggregate shock to be even lower, i.e. to drive the bank into bankruptcy, is given by the normal cumulative function:

\[
P(\Gamma_{t+1} \leq \Gamma_{VaR,t+1} | \sigma_{t+1}) = \mathbb{N}(-VaR) = d_t
\]  

(31)

But as the share of equity holding increases, the ability of the bank to absorb negative shocks increases and its probability of default decreases, so that the cost of unsecured debt decreases and its interest rate premium above the risk free interest rate $r^d - r$ should decrease, which provides an incentive for the bank to choose an adequate share of debt versus equity finance: more equity reduces the cost of debt, but more equity is costly as equity holders require a premium such that Modigliani-Miller does not hold, which leads us to derive the endogenous or market implied Value-at-Risk constraint.
**Endogenous VaR.** I now turn to the case of an endogenous VaR constraint. The bank chooses the variable \( V_{aR_t} \) and maximises its program 24 subject to the balance sheet identity 26 as well as the leverage constraint equation 27, the FOC from the household side equation 22 and the probability of default from equation 31 as computed by market participants.

\[
\max_{V_{aR_t}} \sum_{t=0}^{\infty} E \{ \beta t^t \{ \Pi_t \} \\
\text{subject to:} \quad B_t = D_t + K_t \\
K_t \geq \left( \frac{1 + r^d_t}{(1 + r_B,t)(1 - V_{aR,t})} - 1 \right) D_t \\
r^d_t = \frac{\lambda_t}{\beta E \lambda_{t+1}(1 - E(d_{t+1})) - 1} \\
d_t = N(-V_{aR_t})
\]

The first order condition for the endogenous Value-at-Risk, that is to say the endogenous choice of bank risk on its liability side, or equivalently the optimal mix between uninsured debt and equity, is given by:

\[
\frac{\partial r^d_t}{\partial V_{aR_t}} D_t = \left( (1 - E(a_{t+1}))(1 + r_{B,t}) - (1 + \mu^d_t) \right) \frac{\partial K_t}{\partial V_{aR_t}} 
\]

This equation clearly shows the equality between the marginal cost of having an tighter VaR constraint which implies that the bank bears the net return of the additional capital required to meet the VaR, versus the marginal gain of having an tighter VaR constraint which implies that the bank reduces it cost of unsecured funding. The previous equation writes fully:

\[
\frac{E(\lambda_t)}{\beta \lambda_{t+1}(1 - E(d_{t+1}))} N'(-V_{aR_t}) D_t = \left( (1 + r_{B,t})(1 - E(a_{t+1})) - (1 + \mu^d_t) \right) D_t \\
\left( \frac{E(\lambda_t)}{\beta \lambda_{t+1}(1 - E(d_{t+1}))} N'(-V_{aR_t})(1 + r_{B,t})(1 - V_{aR,t}) + (1 + r^d_t)(1 + r_{B,t}) E_{V_{aR,t}} \sigma_{t+1} \right) \\
\left( (1 + r_{B,t})(1 - V_{aR,t}) \right)^2
\]

### 3.4 Hotelling’s model of deposit allocation

The preferences of the households are modified as follows in order to introduce a rational for the co-existence of several financial intermediaries with different sizes and leverage:

\[
\max_{C_t, L_t, D_t, K_t} \sum_{t=0}^{\infty} \beta^t \left( \ln(C_t - hC_{t-1}) + \Psi \ln(1 - L_t) - \theta^b \text{dist}^b \right)
\]

Deposits \( D^{\text{ins}} \) at the aggregate are capped by the maximum amount the regulator decided to insure, so that total debt is composed of insured and uninsured debt, the ratio of both being determined by deposit insurance.

The allocation of insured deposits across banks is micro-founded using a Hotelling spacial approach. I assume that the representative household is composed of a continuum of members spread uniformly around the unit circle. Thus the utility function includes the term \( \theta^b \text{dist}^b \) where \( b \in \{ \text{sav}; \text{inv} \} \) stands for savings or investment bank; the parameter \( \theta \) corresponds to the cost of travelling to each bank, that is to say it corresponds to the feature of each of the
bank type, explicit or implicit costs such as, possibly, reputation, international presence, array of services provided to the depositor... Conversely the variable dist corresponds to the physical distance of each member of the household from each local bank agency. If there are \( n \) local agencies belonging either to the savings \((n/2)\) or investment bank \((n/2)\), spread alternatively on the unit circle, the distance of a household member from a local branch of the investment bank will be given by \( \text{dist}^{\text{inv}} = \frac{1}{n} - \text{dist}^{\text{sav}} \). A household member will be indifferent between depositing in one bank or the other if \( \theta^{\text{inv}} \text{dist}^{\text{inv}} = \theta^{\text{sav}} \text{dist}^{\text{sav}} \), so that the fraction of household members choosing the investment bank is given by \( 2 \text{dist}^{\text{inv}} = \frac{\theta^{\text{inv}}}{\theta^{\text{sav}} + \theta^{\text{inv}}} \frac{n}{2} \), since a mass \( \text{dist}^{\text{inv}} \) of household members are within reach of the investment bank on each side of the local agency on the circle. Thus each bank receives the deposits raised by its \( n/2 \) number of local agencies:

\[
D_{t}^{\text{ins,inv}} = \frac{\theta^{\text{sav}}}{\theta^{\text{sav}} + \theta^{\text{inv}}} D_{t}^{\text{ins}} \quad (35)
\]

\[
D_{t}^{\text{ins,sav}} = \frac{\theta^{\text{inv}}}{\theta^{\text{sav}} + \theta^{\text{inv}}} D_{t}^{\text{ins}} \quad (36)
\]
4 Numerical Simulation

4.1 Parametrisation

The list of parameters is displayed in Table 1.

The variance of the productivity shock $\epsilon$ as well as the variance of expected tail productivity shocks $\xi$ are key to determine bank holding of equity as it is modelled as a cushion against a larger share of non-performing loans in the tail. Thus more volatility creates more equity holding and vice-versa. To that extent, it allows the model to give a benchmark about how much equity banks should hold absent mispricing frictions if it wants to be resilient in a specific quantile of asset returns.

In addition, the preference parameter, $\psi$, is set to match steady-state hours of work equal to 0.44, as in Kydland and Prescott (1991). Also the wedge that insures that Modigliani-Miller does not hold is set such that the dividend rate matches the yearly return of the STOXX Europe 600 Banks of 9.3% over the period 2000-september 2008; in fact this wedge boils down to having a tax shield on debt or equivalently a larger taxation on equity holding paid by investors, such that the investors want to be indifferent between the net return of debt or equity while banks choose between funding themselves with gross debt or equity. The elasticity of substitution across different varieties, $\epsilon$, is set in order to target a steady state gross mark-up $p_{i,t} = mc_{i,t} \epsilon^{\epsilon - 1}$ equal to 1.5 (Christensen and Dib, 2008). The remaining parameters are standard in the literature, such as the capital share of 0.33 or the discount factor 0.99 to get a yearly interest rate of 4.1%.
Table 1: Parameter choice (calibration when no bailout)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Match</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discount factor</td>
<td>$\beta$</td>
<td>0.99 yearly interest rate of 4.1%</td>
</tr>
<tr>
<td>Preference leisure vs. consumption</td>
<td>$\psi$</td>
<td>0.8 steady-state hours 0.40</td>
</tr>
<tr>
<td>Share of capital</td>
<td>$\alpha$</td>
<td>0.33</td>
</tr>
<tr>
<td>Capital depreciation</td>
<td>$\delta$</td>
<td>0.025</td>
</tr>
<tr>
<td>Recovery rate if default</td>
<td>$\gamma$</td>
<td>0.4</td>
</tr>
<tr>
<td>Deep habits in consumption</td>
<td>$h$</td>
<td>0.595</td>
</tr>
<tr>
<td>Elast. of subst. between varieties</td>
<td>$\epsilon$</td>
<td>3 gross mark-up $\frac{\epsilon}{\epsilon-1} = 1.5$</td>
</tr>
<tr>
<td>Minimum idiosyncratic productivity shock</td>
<td>$a$</td>
<td>0.6 get a share of defaulting firms of 1.3%</td>
</tr>
<tr>
<td>Productivity shock persistence</td>
<td>$\rho$</td>
<td>0.9</td>
</tr>
<tr>
<td>Productivity shock variance</td>
<td>$\sigma$</td>
<td>0.005</td>
</tr>
<tr>
<td>Larger expected pivity shock persistence</td>
<td>$\rho_\xi$</td>
<td>0.9</td>
</tr>
<tr>
<td>Larger expected pivity shock variance</td>
<td>$\sigma_\xi$</td>
<td>0.0005</td>
</tr>
<tr>
<td>tax shield (non M-M wedge)</td>
<td>$\chi$</td>
<td>0.55 9.3% yearly return</td>
</tr>
<tr>
<td>Bank capital adjustment cost</td>
<td>$\phi$</td>
<td>10</td>
</tr>
<tr>
<td>Capital goods adjustment cost</td>
<td>$\kappa$</td>
<td>5</td>
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</tbody>
</table>
### Table 2: Selected business cycle statistics

<table>
<thead>
<tr>
<th>Households side</th>
<th>Production side</th>
<th>Financial side</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable</td>
<td>$\sigma/\sigma_Y$</td>
<td>Variable</td>
</tr>
<tr>
<td>$C$</td>
<td>0.8148</td>
<td>$Y$</td>
</tr>
<tr>
<td>$L$</td>
<td>0.1298</td>
<td>$\sigma$</td>
</tr>
<tr>
<td>$w$</td>
<td>1.3143</td>
<td>$\Pi_e$</td>
</tr>
<tr>
<td>$S$</td>
<td>9.8689</td>
<td>$r_B$</td>
</tr>
<tr>
<td>$D$</td>
<td>9.7373</td>
<td>$B$</td>
</tr>
<tr>
<td>$r_D$</td>
<td>0.1957</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>$K$</td>
<td>0.3168</td>
<td>$\alpha_{VaR}$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.4348</td>
<td></td>
</tr>
</tbody>
</table>

#### 4.2 Summary statistics

**Business cycle statistics**  As a result, I can match standard business cycle statistics (for instance see King and Rebelo (2000)), of which a short selection\(^4\) is displayed in table 2. I report statistics for the case without bailout expectations but with endogenous securitisation due to markets’ misperception of off-balance sheet items.

Total savings $S = D + K$ is more volatile than standard models, with a relative standard deviation of about 9. Also the return on equity $\delta$ is more volatile than the return on unsecured debt $r_D$, which is consistent with the fact that capital is modelled as a capital cushion that absorbs shocks first. In the meantime, unsecured debt is more volatile than bank equity, which again is consistent with the modelling choice of market discipline going through the provision of unsecured debt.

On the production side, bank loans $B$ is very volatile, but actual capital used by entrepreneurs $k$ is slightly less so as they can also rely on the value of undepreciated capital to invest next period. In addition the share of defaulting entrepreneurs $\alpha$ is more volatile than the share of entrepreneurs defaulting at the Value-at-Risk threshold $\alpha_{VaR}$, which stems from the fact that banks may face shocks on the asset quality of their portfolio, but can choose to build a capital buffer to hedge against large share of non-performing loans.

On the financial side, both bank leverage and VaR threshold are more volatile than output. The economy features a share of defaulting firms of 1.38% per quarters, while the bank can sustain a fraction of non-performing loans of up to 1.81% thanks to its capital cushion of 13.61% of its assets.

#### 4.3 Main dynamics of the benchmark model

**Response to a productivity shock.** The benchmark model allows to replicate several key features of banking economics in a unified and flexible framework. The impulse response functions of the model after a one standard error productivity shock is given in figure 3. Let’s first look at the benchmark case in black dashed lines with no regulatory capital, no securitisation and no bailout.

The main cyclical features of the model, after a negative productivity shock of one standard deviation, are:

1. procyclical bank leverage $B/K$;
2. countercyclical bank capital $K/B$;
3. procyclical bank lending $B$;

\(^4\) See Table ?? in the appendix for the whole list.
4. countercyclical borrowing costs $r_B$: as lending increases in periods of boom, lending rates decrease;

5. countercyclical portfolio risk $a$ and $a_{VaR}$: the share of non-performing loans the bank faces or expect to face when it reaches its VaR decreases when the economy expands;

6. procyclical “true” default probabilities $d$: the default probability of the bank decreases when it faces a good shock.

Response to a larger expected shock. Next, I provide the impulse response functions for a “volatility” shock in figure 6. This type of shock arise specifically in my model, and can be understood as a shock on the expected size of tail risk. As the model is linear, it is not a proper volatility shock, but allows to study the steady state dynamics about the expectations of the size of the tail of the distribution. It is akin to a productivity shock but which does not materialize at the steady state and as such could be better understood a shock on the expectations of productivity shocks.

This shock on the size of the expected tail productivity shock has the following consequence. In the benchmark case (dashed black line), when the economy faces a larger expected shock, it is detrimental to the economy as a larger capital cushion is required by market forces which creates an endogenous procyclicality compared to the models that consider capital only as a constant requirement. Thus as the bank is forced to increase its equity holding $K$ to cope with a deteriorate Value-at-Risk $VaR$ and recalling that equity is costly due to the tax shield on debt, the bank has to reduce its leverage $B/K$ and cut on lending to entrepreneurs $B$. But as the bank cannot adjust immediately its capital position, its expected probability of default $d$ rises and the risk premium paid on debt increases which further reduce the availability of outside unsecured finance to the bank. It further deteriorates the economy.

Nevertheless, firms production $Y$ decreases with a delay as it can rely on the value of its non-depreciated stock of capital goods which can be sold on the second hand market to finance part of the production next period. Thus a shock in the resilience of the banking sector does not spread immediately to the real economy: bank loans $B$ drop sharply while actual firms capital stock $k$ decays gradually. In other words, a shock on the expected resilience of the banking sector has a long lasting and delayed effect on the real economy, while a standard productivity shock has an immediate effect: shocks originating in the banking sector and affecting their resilience are potentially more harmful than standard productivity shocks.

5 Conclusion

The present model provides a theory of bank capital subject only to market forces, with default risks partly internalised via expectations. The present approach assumes that agents form expectations not only on the mean future state but also on the whole distribution of shocks. Since Modigliani-Miller does not apply in this framework, banks make a non-trivial choice on the liability side of their balance sheet. An endogenous Value-at-Risk constraint arise, whereby the bank can save on its expensive equity but at the cost of a larger risk premium on unsecured debt. This gives rise to procyclical bank leverage, an empirical fact that did not hold in previous DSGE models that typically feature procyclical credit cycle but countercyclical leverage.

This endogeneous Value-at-Risk is inherently procyclical as a negative productivity shock shifts the expected distribution of bank portfolio shocks to the left and thus makes subsequent losses more likely. Larger current and expected non-performing loans push for more equity holding, thus increases average bank funding costs and thus lowers banks’ lending. In the meantime, larger banks are more subject to this procyclicality as they leveraged more in the first place by relying on a larger pool of deposits and thus a cheaper access to unsecured debt. The heterogeneity in bank size in my setting is not due to the regulation nor ad-hoc
assumptions about their specialisation but by technological constraints (Hotelling spacial model with travelling costs to local bank agencies) that made some banks more widely available to capture deposits. Only in a second step are larger banks able to benefit from a cheaper access to outside finance as the fixed cost of entering the world markets are smaller per unit of asset.

In terms of policy implications, results suggest that (i) prudential regulators should be worried about the perception of market participants as to the resilience of financial institutions as expectations of fatter tail risks can generate an endogeneous process of bank deleveraging reducing the loan supply to the economy; (ii) countercyclical capital regulation can make sense if market participants have excessively negative tail risk expectations in order to relax the leverage constraint of banks. However it is unclear that banks would react to a looser regulatory leverage ratio in bad times as leverage is inherently procyclical, making a regulatory leverage ratio non-binding in case of downturn.

The steady-state expectations of tail risk can possibly be calibrated to match systemic risks measures. Thus this setting gives the possibility to study the ability of capital regulation to efficiently internalise risks with respect to the market equilibrium, but also its potential distortionary impact when misdesigned. It can also be used to analyse the effects of off-balance sheet practices or bailout expectations that would distort the expectations of tail portfolio shocks and thus the lower perceived risk would allow the bank to increase its leverage, raising its actual default probability. These financial stability implications are analysed in a subsequent companion paper.

6 Appendix
Figure 2: IRFs after a one standard deviation negative productivity shock
Figure 3: IRFs after a one standard deviation negative productivity shock

- C (% SS)
- E(C) (% SS)
- C|default (% SS)
- Y (% SS)
- D (% SS)
- K (% SS)
- B (% SS)
- k (% SS)
- VaR (% SS)
- d (% SS)
- Mispricing Gap (% SS)
- K/B (% SS)
- abar (% SS)
- abar_{VaR} (% SS)
- sec (% SS)

with distorted risk weights
correct risk weights
benchmark: no regulatory capital, no securitisation
Figure 4: IRFs after a one standard deviation negative productivity shock
Figure 5: IRFs after a one standard deviation increase in expected size of aggregate productivity shocks
Figure 6: IRFs after a one standard deviation increase in expected size of aggregate productivity shocks
Figure 7: IRFs after a one standard deviation increase in expected size of aggregate productivity shocks.
References


