An Application of Behavioral Welfare Economics to Consumption Data
Preliminary Version

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Abstract

Revealed preferences are a potentially powerful tool for conducting welfare analyses, but due to forces such as behavioral biases, "irrational behaviors" such as non-transitivity (i.e. cycles of preferences) can appear in revealed preferences, which render them useless for welfare purposes. In this work, we provide the first empirical application of a leading proposal for welfare analysis in the presence of behavioral biases, presented in Bernheim and Rangel (2009), called the *Strict Unambiguous Choice Relation* (hereafter Strict UCR). However, there are two potential problems in the application of this approach: Strict UCR can be cyclic on consumption data, which are incomplete, and Strict UCR is very conservative, which may make it overly coarse, and thus, useless (Rubinstein and Salant 2012). We address these concerns using a dataset of package grocery purchases by 397 households over a two years period in the United States. We find that the Strict UCR relation is cyclic on incomplete data[1], but much less than revealed preferences (8% against 99% of agents exhibit cycles). It is also almost complete, as in our application, almost all bundles are comparable. In addition, we introduced two new relations based on the idea of the Strict UCR, but that guarantee acyclicity. One is shown to be equivalent on complete data, and the other is in the same flavor than Strict UCR, but is more conservative.

Keywords: Behavioral economics, Economic welfare, Strong Axiom of Revealed Preferences, Scanner data
Jel Classification: D60, D110

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[1]Complete data: having a choice in all possible subsets of alternatives

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Introduction

Bernheim and Rangel (2009) introduced "Behavioral Welfare Economics" with the aim to build theoretical foundations to welfare inferences performed in behavioral economics. One traditional approach to welfare analysis is revealed preferences, introduced by Samuelson (1938) and applied to finite choices by Afriat (1967) and Varian (1982). This method assumes that observed choices are the correct ones, and that agents (here individuals) maximize their preferences (or utility), depending on the primitive of the model. However behavioral economics has shown that individuals do not maximize a welfare order, but use instead heuristics to choose. Thus observed choice might not be the welfare maximizing ones.

These results lead behavioral economists to build *ad hoc* models to perform their welfare inferences. However, they tend to rely on the intuition of the modeler to assess the real welfare order, but misspecification of the model leads to incorrect welfare inferences. In these models, the observed choice might not be the best one, and might actually contradict the preference order. Salant and Rubinstein (2008) and Rubinstein and Salant (2012) formalized this approach, but before that, theoretical foundations were often unclear. Bernheim and Rangel is another attempt to take into account these heuristics, using the concept of frame as introduced by Tversky and Kahneman (1981).

Bernheim and Rangel's main theoretical stance is to keep the observed choice as the real preference order, because as modelers, it is usually the only data we have access to. However, they introduce the concept of *ancillary condition* (a kind of frame), to take into account heuristics. From a welfare perspective, in their model, choices can be compared only if they are affected by the same ancillary condition. The modeler can even ignore in the welfare inference some choices, if he has good reasons to believe they are not the right ones, but it cannot reverse the observation that are "welfare relevant". The reason to ignore an observation is essentially that the framing is considered as too important by the modeler.

The model considers choices from menus, which does not need prices (and is in this sense more general). We adapt it to our empirical purposes by introducing prices, but the theoretical results of section 2 do not rely on their existence. Of the four relations introduced in Bernheim and Rangel (2009), we are interested in the Strict Unambiguous Choice Relation, which in words can be described as follows (x and y are two alternatives):

> x is Strict UCR y if and only if whenever x and y are both available, x is chosen at least once and y is never chosen.

Bernheim and Rangel show that this relation extends the classical revealed preference relation in a very nice way. In particular, it keeps the first and second theorem of welfare, with the caveat that the transfers are defined a bit differently.

For this work, we simplified their assumptions, and in particular we assume that a consumer choosing in a grocery store always has the same ancillary condition, so that all observations we have are "welfare relevant". We are also only interested in the relation presented above and not in the three others, because it is the only one that guarantees acyclicity of preferences, and thus is interesting for welfare inference. A second reason for that selection is the context of our data: consumer data with prices gives almost surely (in a probabilistic meaning) strict relation, and no indifferences. Thus in our context, the general and strong axiom of revealed preferences as presented by Afriat (1967), Afriat (1973), and Varian (1982) are the same.

We have divided this paper in three part. In section 1, we apply Strict UCR to consumption data[2]. We show that because one essential assumption to get acyclicity, complete data, is violated, Strict UCR is cyclic. A dataset is said to be *complete* if we observe at least one choice in all (nonempty) possible subset of alternatives[3]. However we also show that on this dataset, the preference relation obtained is almost complete, whereas we observe much less cycle of preferences than with traditional preferences.

This mixed results lead to section 2: willing to capitalize on the results of Strict UCR, we introduce two new relations, UPR and Strong UPR. We prove that UPR is the natural counterpart of Strict UCR in the state of preference relation (as opposed to the state of choice sets) which guarantees acyclicity. On the other hand, Strong UPR follows the same kind of conservative reasoning as Strict UCR, but is not equivalent. We provide axioms to justify its foundations and we show that it is more conservative than the two other ones. We also show how it is different from the transitive core as defined by Nishimura (2014).

Finally, section 3 show the application of the two new relations to our dataset. We show that in fact the preference relation we got is almost complete, and acyclic. The latter means that we can perform a meaningful transitive closure on the preference relation. This transitive closure still does not guarantee completness. The quasi-completness means that in general we are able to perform welfare inferences (but not always, as some bundles are incomparable).

1 Application of the Strict UCR relation

We applied the Strict UCR to consumption data from the Denver area, between February 1993 and February 1995. Before describing more precisely the data, let us define properly the Strict UCR relation in our context. In the situation of consumption data, alternatives x and y are bundles. Let us association to them prices p_x and p_y. x is *available* in the situation where y is chosen if and only if $p_y x \leq p_x x$. In the situation where all bundles and all prices are different (which is the case in our data), it means that:

$$xP^*y \Leftrightarrow \begin{cases} p_x x > p_x y \\ p_y y < p_y x \end{cases} \qquad \text{(Strict UCR)}$$

Where xP^*y is used to say that x is strictly unambiguously chosen over y. In words, the condition belows means that y should be available when x is chosen, whereas x should not be available when y is chosen. If y was not available, then it would mean that x and y are not comparable. Whereas if x was available when y is chosen, the choice is ambiguous and thus will not generate a preference relation.

[2]The panel data we used is over purchases of grocery packages in the United States. It was first used in economics by Aguiar and Hurst (2007). Our sample has 397 households, over two years, its more precise description is in section 1.

[3]This is a very demanding assumptions, in all generalities, the number of such subsets for a set of size n is $2^n - 1$ (or $2^n - n - 1$ if singletons are excluded, as choices are obvious in that situation).

1.1 The Data

This data was originally used in economics in Aguiar and Hurst (2007). The data used here is close to the treatment Dean and Martin (forthcoming) have used. The data was collected by the marketing firm ACNeilson. The full description of the original dataset, and in particular its demographic characteristics, can be found in the latter, whereas a description of the more precise sample used here is available in the former. Essentially, we restrict for the 2-week period to 397 households of the Denver metropolitan area, between February 1993 and February 1995 (the original data counts $2,100$ households). This sample is a balanced panel over the period, thus avoiding the problems of repeated cross-sections in terms of comparability. ACNeilson is a marketing firm, and they attached great importance to the reliability of their data. Thus households document the UPC (universal product code) of the products they buy at home, regrouping all shops they have been in. Thanks to the UPC, prices and characteristics of the goods at the moment of the purchase are recovered. This procedure guarantees a quite good sample, even though neither the representativity nor the reliability really matters here, as we want first to test the Strict UCR on real data. The aim here is a proof of concept: does the method work? The fact that the sample is better than most of what is available in the literature is a bonus, showing that indeed, households are close to rationality.

For comparability reasons, goods are regrouped into 36 categories (the full description of the categories is in appendix 4.1). Products that do not fall in this classification are excluded. We restrict our attention to a 24 month periods, which means 48 two-weeks periods. The characteristics of the data are the following: 397 households, with an average of 18.1 purchases, 3.5 store trips, and \$20.4 in expenditure per period. The demography of this sample is similar to the restricted one and available in table 4.1 at page 20, in the appendix.

The aggregation over time is to alleviate concerns about storage of some items (say, pastas or rice). However, aggregating on long periods reduce the number of observations and the probability of violations. That is the main reason why choose to aggregate over two weeks and not a month or longer[4]. Price indices are constructed for this periods, using the classical method in the literature of "Stone" prices:

$$P_{Jt} = \sum_{i \in J} w_{it} p_{it}$$

where P_{Jt} is the price index for good category J in the period t, w_{it} is the budget share for UPC code i in the period t, and p_{it} is the mean price for UPC code i in period t. Dean and Martin (forthcoming) examines also other indices and do not find significant differences, thus we keep that one and do not investigate more.

Here of course, not all items a households can buy are represented, the average sum used per period is here to testify that. For instance, leisures are not in our sample. To get acyclicity from utility maximization on a restricted sample of all possible purchases, the assumption of additive separability between the goods considered and the others is required. It is quite strong, but also standard in revealed preferences testing. It is also

[4]The simulations have also been performed on a monthly time period and with only 3 aggregate goods. The results are expected: less violations, but still in line. We choose that case as it is more stringent and thus more interesting for our proof of concept.

assumed that all households faces the same prices in a given period. This assumption is necessary, because it might happen that a household does not buy a category of good in a given period, thus revealing no prices for that category and hampering revealed preferences testing. It could be possible to keep only households that have bought all types in all periods, but it would restrict too much the sample[5]. Of course, in reality, prices vary between stores and in a given period, we thus introduce an error, but it is standard and the error is likely to be small (goods prices in a two-week time period will not double).

The last important assumption made for the empirical testing is that preferences are stable over time. We need this assumption to be able to perform comparison between the first and the last period. If preferences were to change, then having a violations of revealed preferences would only mean that preferences have changed and would not be informative *per se*.

One important feature of the data we have is that two bundles never comes twice for the same household (relations are built on a household level). The prices are also always different from one period to the other, thus allowing the use of the definition of Strict UCR.

1.2 Results

Using the definition provided in equation Strict UCR, we test the Strict UCR relation on the data described above, and compared it to the traditional revealed preference relation:

$$xRPy \Leftrightarrow p_x x > p_x y \tag{1}$$

Where $xRPy$ is used to denote that x is revealed preferred to y. Because prices are always different from one period to the other, we can use the strict inequality. In our context, the general and strong axiom of revealed preferences are the same (reason why we have used the strict inequality in the definition of P^*).

We apply revealed preferences and Strict UCR to the data, and we obtain the following results: over the 397 households, *30* exhibit cycles (7.6%) of Strict UCR whereas only 4 do *not* exhibit cycles of revealed preferences. The characteristics in terms of relations per bundle per agent and in terms of relations removed to rationalize the data are presented in table 1.1.

This table tells us that the Strict UCR performs much better than revealed preferences. Indeed, in most cases it is acyclic. The second surprise is that contrary to what Rubinstein and Salant (2012) were afraid of, the Strict UCR relation is almost complete. Admittedly in this setting, applying it is equivalent to removing cycles of size 2, as two bundles are compared in only two situations. However, one main drawback is on that common empirical setting, acyclicity is not guaranteed. This leads to the second section of this paper, where we define two new relations and relates them to the Strict UCR. It should be noted that until now, we have only considered directly revealed preferred relations, and have not considered any kind of transitive closure. We will come back to this point at the end of the next section. The reason is simple: in our experimental results, we have

[5] Dean and Martin (forthcoming) found that it would reduce the sample size by 85% for the monthly sample; In our case, it would be even more stringent.

	Average	Median	Std Deviation	Minimum	Maximum
Revealed Preferences	46.60	46.79	0.37	45.21	47.54
Strict UCR	46.01	46.16	0.34	44.79	46.75
Cycles of RP	476.79	12	6,067.07	0	113,489
Cycles of Strict UCR	0.2	0	1.1	0	18

Table 1.1: Average number of relation per bundle, the theoretical maximum is 47 (each bundle of an agent should compare to all others) and Statistical description of the number of cycles per household. It should be noted that the average number of cycles for a cyclic agent of Strict UCR is 2.

cycles of preferences. In this situation, closing transitively is meaningless: we need first a preference relation that is acyclic. Something we devote the next section to.

2 The (Strong) Unambiguous Preference Relation

The previous section shows interesting results: the Strict UCR is a clear improvement over the classical revealed preferences. However, as expected from the fact that real data does not verify the complete data hypothesis, the Strict UCR does not guarantee acyclicity. This section is dedicated to the solution of this problem from a theoretical point of view, by introducing two new relations, the *Unambiguous Preference Relation* (hereafter UPR) and the *Strong Unambiguous Preference Relation* (hereafter Strong UPR). We show the equivalence of UPR to with Strict UCR on complete data, and in fact a slightly more general result. Strong UPR will differ from the Strict UCR, but follow the same spirit, as we will show later, and is in fact more conservative (that is, less complete) than the other two. To understand the relations, we introduce some notations and definitions and proves two theorems showing their relevance.

2.1 Definitions and Notations

The Strict UCR is defined in the very general choice set framework. UPR and Strong UPR are in contrast defined in the (binary) relations framework. We do not think we really loose generality in the process, as we can always build relations from observed choice, for instance by using revealed preferences.

Define X as the grand set of all alternatives and x an element from that set[6]. A and B will be in all the following nonempty subsets of X. To build our preference relations, we need to work on the space of binary relation on X, which is $X \times X$.

Definition 1 (Binary Relation)**.** *For an arbitrary set X, a binary relation on X is a subset of $X \times X$ with a generic notation R.*

For two elements $x, y \in X$, we write that xRy to mean that $(x, y) \in X \times X$.

From now on, x, y and z will represent three distinct elements of the set X. In our case, R is a relation of preference, so that xRy means that x is preferred to y.

[6] In the previous section, x was a bundle, but here, so that X is the (infinite) set of all possible bundles, but X is a very general set of alternatives

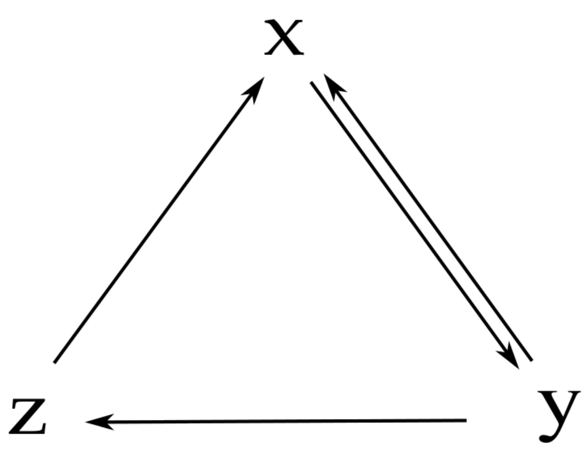

Figure 2.1: Cycles and subcycles of preferences, where the tail of the arrow is the preferred alternative, that is, zRx. Here $xRyRzRx$ is a cycle and $xRyRx$ is a subcycle of $xRyRzRx$.

Definition 2 (Characteristics of binary relations)**.** *R is said to be:*

- *reflexive if* $\forall x \in X$*,* xRx*;*
- *complete if* $\forall x, y \in X$*,* xRy *or* yRx*;*
- *transitive if* xRy *and* yRz *implies* xRz*;*
- *R is antisymmetric if* xRy *and* yRx *implies* xIy *(I is used to denote indifference in preferences).*

A complete, transitive and antisymmetric binary relation is a total order. In terms of preferences, it means that they are well-defined and that we can build a utility out of it, thanks to Afriat's theorem.

Definition 3 (Antisymmetric part of a binary relation)**.** *The antisymmetric part of a binary relation is denoted by* P *and is such that:*

$$xPy \Leftrightarrow xRy \wedge \neg yRx$$

It means that x is strictly preferred to y.

Definition 4 (Symmetric part of a binary relation)**.** *The symmetric part of a binary relation is denoted by* I *and is such that:*

$$xIy \Leftrightarrow xRy \wedge yRx$$

It means that the individual is indifferent between x and y.

Binary relations can also be seen in terms of a (directed) graph, where X is the set of vertices and $X \times X$ is the set of edges. Here, for clarity, we keep the economic denomination, so that X is a set of alternatives, whereas $X \times X$ is a set of preference relations.

Definition 5 (Cycle). *A cycle of preference relations R is a finite sequence $(x_i)_{i=1}^n$ of distinct alternatives such that $x_i R x_{i+1}$ for all $i \in \{1, \dots, n-1\}$ and $x_n R x_1$, with at least one preference relation being strict (that is, P instead of R).*

$\mathcal{C}(X)$ is the set of all cycles on a set of preference relations $X \times X$.

From now on, for notational brevity, we will denote the cycle (x_i) and $x_{n+1} = x_1$. The latter is possible because in a cycle, the starting point does not matter.

In the absence of any strict preference relation in a cycle, we may argue that the individual is indifferent between all choices, reason why we need strict relations. If you keep in mind the experimental data we have, as we only obtain strict relations, it is not really a problem.

Definition 6 (Subcycle of a Cycle). *A subcycle is a cycle that is strictly smaller than another cycle (that is, contains strictly less alternatives) and shares at least one preference relation.*

A cycle and a subcycle are represented on figure 2.1.

2.2 Relation between choices and preferences

Another important set of definitions to understand this paper is related to choices. We introduce here some elements necessary for the proofs of our results.

Definition 7 (Choice function). *A choice function is a function:*

$$c : 2^X \to X$$
$$A \to x$$

with $A \subseteq X$ and $c(A) \in A$.

That is, it is a function from the set of subsets of X, that associates to a subset A an element of that subset.

$c(A)$ is interpreted as the (observed) choice of an agent from the choice set A.

Definition 8 (Choice correspondence). *We call a choice correspondence a function:*

$$C : 2^X \to 2^X$$
$$A \to C(A)$$

with $A \subseteq X$, $C(A) \subseteq A$ and $C(A) \neq \emptyset$.

That is, a function from the set of partitions of X that associates to a subset A a (nonempty) subset of A.

$C(A)$ is interpreted as the (observed) choices of an agent from the choice set A. All choice functions are choice correspondences, but the opposite is not true. The difference between a choice function and a choice correspondence is that the choice function only associates one element to a set of alternatives, whereas the correspondence may associate several. If we interpret the subset A as a choice situation, where an agent chooses between several alternatives, it means that an observer observes different choices in the same situation.

We define revealed preferences and Bernheim and Rangel's Strict UCR in this (more general) context.

Definition 9 (Revealed Preferences)**.** *For choice correspondence, we say that, for two different elements x and y, x is revealed preferred to y (denoted $xRPy$) if and only if:*

$$\exists A \subseteq X : x, y \in A, x \in C(A), y \notin C(A)$$

The definition is the same for choice functions, except that we do not need the last condition $y \notin c(A)$ as $c(A)$ contains only one element. Revealed preferences are a way for an (external) observer to assess the preferences of an agent. It essentially says that x is revealed preferred to y if we observe from a set where both x and y are available that x is chosen (and y is not). The preferences can be considered as weak or strict. The use of weak preferences allow for indifferences. However, on empirical applications, the relation is almost always assumed to be strict. Indeed, if we consider observed data, it is not possible to determine if an individual is indifferent between two choices or not. The use of weak relations would also generates problem if we had cycles of size 2 (which indeed we have), as an agent would be indifferent between those two elements. The definition we used in section 1 is a particular case of the case of choice function[7].

Revealed preferences and preferences in general are binary relations on the set of alternatives. From now on, we will denote $B_{RP}(X)$ the set of binary relations generated by revealed preferences on the set of alternatives X.

Axiom 1 (Strong Axiom of Revealed Preferences)**.** *Revealed preferences on the set of alternatives X verifies the* SARP *if and only if the set of binary relation generated by revealed preferences on X does not exhibit cycles.*

When the axiom is verified, we know from Houthakker (1950) and Afriat (1967) that it will be possible to build a utility function from the observations (under the condition of completeness, that is, all alternatives are directly or indirectly comparable).

Now, let us introduce formally the general definition of Strict UCR.

Definition 10 (Strict Unambiguous Choice Relation)**.** *x is said to be strictly unambiguously chosen over y (xP^*y), if when x and y are available, x is sometimes chosen and y is never chosen:*

$$xP^*y \Leftrightarrow \forall A \subseteq X,\ \ x, y \in A, y \notin C(A) \wedge \exists A\ x \in C(A)$$

The definition with choice functions is the same. This is almost like revealed preferences, and it does not overrule an observed choice, contrary to what Rubinstein and Salant (2012) are advocating for instance. The main difference with revealed preferences is that you have to consider all choice situations together to retrieve preferences, whereas you can consider them one by one for revealed preferences. The interest of considering the whole data at hand will soon appear, after we define what is complete data.

Definition 11 (Complete Data)**.** *A dataset is said to be complete if we have a choice for all subset of the set of alternatives X. That is:*

$$\forall A \subseteq X, A \neq \emptyset, \exists C(A) \neq \emptyset$$

[7]Because all prices are different in our application, so that a we only observe one choice per situation.

The definition is the same for choice functions. Similarly to what we have done with revealed preferences, we will denote SUCR(X) the set of binary relation generated by the strict unambiguous choice relations on the set of alternatives X. This definition gained a great traction in the literature and the rationale behind it is justified in Bernheim and Rangel's article. It should be noted that SUCR(X) $\subseteq B_{RP}(X)$[8].

Now, we have the first interesting result, coming from Bernheim and Rangel's corollary one.

Proposition 1

With complete data, the strict unambiguous choice relation is acyclic.

An important remark about this proposition is that having complete data is very difficult in real-life applications. Indeed, the number of subsets for a set of size n is $2^n - 1$[9].

However, acyclicity is attractive, even though the hypothesis of complete data is essential for the result. Let us illustrate that fact on a simple example (coming from Bernheim and Rangel's example 1):

Example : Consider a set of three alternatives $\{a, b, c\}$, with the following observations from an observer:

- $C(\{a, b\}) = a$
- $C(\{b, c\}) = b$
- $C(\{c, a\}) = c$

Here, the data is incomplete and we have the relations:

- aP^*b
- bP^*c
- cP^*a

Which yields to a cycle of the strict unambiguous choice relation: $aP^*bP^*cP^*a$. This is because the modeler does not have a choice in the set $\{a, b, c\}$. If that was the case (imagine without loss of generality that it is a), then one of the preference above would not appear (here cP^*a) and there are no cycle in the set of binary relations.

The cyclicity problem on incomplete data and the will to apply the strict unambiguous choice relation led us to think another definition that would be applicable. The insight of the definition is that we want acyclic preferences to satisfy the strong axiom of revealed preferences. The idea is essentially to consider all binary relations obtained through revealed preferences and then focus on the cycles and get rid of them. We say that a relation *breaks* a cycle if by removing this (binary) relation, the cycle (of preferences) disappear. Let us define a new set of cycles, the set of cycles that we have to remove to get an acyclic preference relation (thus allowing both the construction of a utility and the transitive closure).

[8]This result is quite straightforward: to have aP^*b for two elements $a, b \in X$ we must have at least a set where a and b are available and where a is chosen but b is not. From this set, we have that $aRPb$.

[9]Or $2^n - n - 1$ by excluding singletons, where the choice is obvious.

Definition 12 (Cycles to be Removed (CR)). *The set of Cycles to be Removed of a set of preference relation $X \times X$ (noted $\text{CR}(X)$) is the set of cycles without subcycles and of cycles that are smallest subcycles of another one.*

Note that there can be more than one smallest subcycle for a given cycle. To avoid this problem of unicity, we will remove all of them, but that is not the only solution[10].

Proposition 2

Removing the relations in $\text{CR}(X)$ guarantees acyclicity.

Proof: Consider a cycle, we want to break it. There are two possibilities:

- Either it has no subcycle, except itself, and it is remove.
- Or it has some subcycle, and in particular it must have at least one smallest subcycle, which is removed. By removing it, as by definition they share at least one preference relation, the cycle is broken: it is not a cycle anymore.

Thus showing acyclicity. ■

The result of the previous proposition leads to the first proposition of definition for UPR.

Definition 13 (Unambiguous Preference Relation (UPR)). UPR *is defined as set of preference relations obtained by revealed preferences, minus the the of cycles to be removed:*

$$\text{UPR}(X) = B_{RP}(X) \backslash \text{CR}(X)$$

With the important note that, to avoid uniqueness problem, we remove *all* smallest subcycles of a cycles. Other definitions are possibles for UPR, but we retain this one, as it closely match the setup until now[11]. At this point, one may argue that we are discriminating in the relation that we keep and remove in our setting, and that somehow as a modeler, if we only had access to the revealed preference information, we should not choose between the cycles to remove. That would agree in spirit with Bernheim and Rangel's stance on observed choices. In the presence of a cycle, there is no maximal element, and there are *a priori* no reason to choose to remove some relations over the others. This leads to the last definition, Strong UPR.

Definition 14 (Strong Unambiguous Preference Relation (Strong UPR)). *Strong* UPR *is defined as the set of preference relations obtained by revealed preferences ($B_{RP}(X)$), minus the set of all cycles ($C(X)$):*

$$SUPR(X) = B_{RP}(X) \backslash C(X)$$

[10] Thus we do not get any kind of minimality, in the sense of the minimum number of cycles to remove to get an acyclic relation, or the minimum number of relations to remove to get an acyclic relation.

[11] Here is an example of another definition, algorithmic:
Consider the set of all revealed preferred relations.

1. At step one, remove all cycles of size 2 from the original set.
2. At step two, consider now the set of remaining relations after step one. Remove all cycles of size 3 from that set.
3. and so on until step $n-1$ (if there are n alternatives).

This proposition is clearly acyclic, and is in fact close (but different) from the original UPR.

The latest definition needs some comments: the idea here is that when a cycle is observed, we assume that the modeler cannot say anything about the preferences of the agent. This might be seem as a rather radical stance. In fact, as our empirical results will show, that is not really extreme. Another important comment is that we might think that this exhibits some kind of equivalence to Strict UCR. We will that in fact, that is not the case, but this relates more closely to the transitive core as proposed by Nishimura (2014).

2.3 Link between UPR, Strong UPR, Strict UCR and the literature

We first characterize the link between UPR, Strong UPR and Strict UCR. Then we examine the link with other preference relation developed in the literature and in particular the transitive core.

One important theorem to explain the choice we made for UPR is the following:

Theorem 1 (*Class of Equivalence*)

A relation willing to be equivalent to the Strict UCR *on complete data with choice functions should remove cycles of size 2 of revealed preferences and only them in that context.*

Proof: We will demonstrate that cycles to be removed of revealed preferences with these assumptions are of size 2, that is, contains only two different alternatives. In the case of cycles of size 2 for revealed preferences, P^* will not generate binary relations, because it means that the choice is ambiguous: there exists two subset $A, B \subseteq X$, such that $a, b \in A \cap B$, with $c(A) = a$ and $c(B) = b$. Therefore we have neither aP^*b nor bP^*a.

Now let us demonstrate that indeed, cycles to be removed of revealed preferences are of size 2. Assume a cycle of size $n > 2$. We will show that it exhibits at least one subcycle of size 2. So that all cycles removed are of size 2.

Let us call $x_1, \ldots, x_n$ the elements of the cycle, in that order. Let us consider the set $\{x_1, \ldots, x_n\}$. We know that we can consider this set, from the assumption of complete data. We also know that there is a choice x_i in that set. Without loss of generality, we can consider $x_i \neq x_1, x_n$, because the starting point of the cycle is arbitrary. Because we are in a cycle, we know that $x_{i-1}RPx_iRPx_{i+1}$. However, from the assumption of choice function, we also have from the choice in the whole set, that x_iRPx_{i-1}. So that we have a cycle of size 2, and the cycle contains a subcycle. It means that all cycles to be removed are of size 2.

Now let us prove that the Strict UCR will only remove those relations. Two cases are possible to have no binary relations in SUCR(X) for two elements a and b:

- Either both a and b are available and never chosen. Therefore there is no revealed preference nor strictly unambiguous choice.
- Or, we must have some sets A and B such that $c(A) = a$ and $c(B) = b$, with $a, b \in A \cap B$. From A, we have that $aRPb$ and from B, we have that $bRPa$. So that we have a cycle of revealed preferences of size 2 between a and b, which will not be in the equivalent relation.

■

Corollary 1

UPR *is equivalent to Strict* UCR *on complete data with choice function.*

Proof: From the previous proof, it is clear that in any cycle, we will find a subcycle of revealed preferences of size 2, or will be itself of size 2. Thus only cycles of size 2 will be removed by UPR. ■

The theorem is in fact broader than showing the equivalence of Strict UCR and UPR in that specific context. It defines the set of all possible equivalent relations to Strict UCR with complete data and choice function. It also means that the alternative definition of footnote 11 is also equivalent to Strict UCR. Another consequence is that Strong UPR will not be equivalent to Strict UCR in that setup, because it will remove all cycles, and in general cycles will be of size larger than 2.

It is important to demonstrate the equivalence in that case, because as I have shown in the example 2.2, SUCR(X) may exhibit cycles when the data is incomplete. It means that it cannot be the case that SUCR(X) = UPR(X) on incomplete data, from the definition, UPR(X) is acyclic, whatever the type of data at hand. Now the reduction to choice function might be more problematic from a theoretical point of view, but from an application point of view, that is not the case. We never have to use choice correspondences on grocery store data, because we observe the choice of one basket of goods from all the goods available. Choice functions are almost used in empirical settings. It means that where they need to be equivalent, these two definitions are.

Corollary 2

On a set of alternatives X, with incomplete data and choice functions, the set of preference relations generated by the Strict UCR *contains the* UPR *and the inclusion might be strict.*

$$\text{UPR}(X) \subseteq \text{SUCR}(X)$$

Proof: UPR$(X) \subseteq$ SUCR(X). Remember first that $SUCR(X) \subseteq B_{RP}(X)$. Two cases are possible to have no binary relations in SUCR(X) for two elements a and b:

- Either both a and b are available and never chosen. Therefore there is no revealed preference nor strict unambiguous choice.
- Or, we must have some sets A and B such that $c(A) = a$ and $c(B) = b$, with $a, b \in A \cap B$. From A, we have that $aRPb$ and from B, we have that $bRPa$. So that we have a cycle of size 2 between a and b, and we have neither $aUPRb$ nor $bUPRa$.

■

This result means that to guarantee acyclicity, we have to give up some preference relations, and in particular the preference relations involved in cycles of the Strict UCR. Indeed, UPR and Strict UCR are equivalent when the experimental data is acyclic. The intuition for this result is the following: when Strict UCR is acyclic (more than 90% of the time in our case), it is possible to virtually extend the choice data to get complete data, while respecting what we have observed before. The completion is quite simple: for a subset of alternatives, if there is a subset where a choice has been observed, use that choice (if several has been observed, choose the maximal one, as given by the preference relation[12]). If you have a subset where no subset has ever been used, any choice will fit, as it cannot contradict what has been previously observed. That way it is possible to complete the choice data, without creating new cycles, while keeping the already observed relations. By the theorem, UPR will be equivalent to Strict UCR on that set, and therefore on the smaller subset of preference relations.

We have shown above that the equivalent to Strict UCR is UPR, and that in fact on incomplete data, to guarantee acyclicity and thus be able to close transitively and construct a utility function out of the data, one correct equivalent is UPR.

2.4 Strong UPR

Now let us turn to Strong UPR and give some reasons to use it :

- It is close, but different from the transitive core of Nishimura (2014).
- This proximity allows for an axiomatization of the relation, close from the revealed preference axiom.

2.4.1 Difference between the Transitive Core and Strong UPR

Before going further in this section, we have to define the transitive core, as used by Nishimura (2014).

Definition 15 (Transitive Core)**.** *The Transitive Core is a set of binary relation. A binary relation xRy is in the transitive core if and only if all others element verify transitivity. Formally:*

$$x\,core(R)y \Leftrightarrow \forall z \in X \begin{cases} zRx \Rightarrow zRy \\ yRz \Rightarrow xRx \end{cases}$$

The idea behind it is that fir a preference relation to be useful for welfare analysis, it has to be transitive. Nishimura assumes the preference relation to be complete, but there is *a priori* no reason why it should not be used on incomplete relations. The transitive core is thus acyclic, and in a sense might be close to Strong UPR. However it is different, as the example in figure 2.2 shows. Straight lines represents indifferences. In terms of cycles, we have cycles $xyzu$, xyu and xyz, because we have only one strict preference xPy. We have to consider complete binary relations. Strong UPR removed all relations that are in the 3 cycles, leading to acyclic preferences. What is left by the transitive core is a bit larger. The preference relation zRu is kept here, because xRz and xRu is true, and yRx and yRu is also. The same is true when we consider the weak relations in th other direction. This feature arise because the transitive core essentially uses three wise comparisons, whereas Strong UPR considers everything. This can be seen in Axiom 6 of Nishimura. It is the only axiom of the transitive core that Strong UPR does not verify.

[12] There might exists several maxima, if the relation is incomplete, then any maxima will fit. It will just create new relations, but not go against what has been observed.

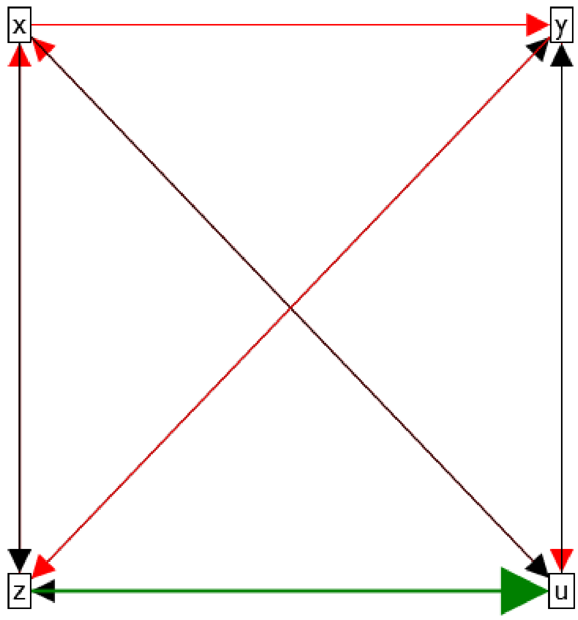

Figure 2.2: The difference between the transitive core and Strong UPR. In red is what would be removed by the transitive core. In Red and green is what is remove by Strong UPR.

2.4.2 Axioms for Strong UPR

The similarity with the Transitive Core suggests that Nishimura's axioms may be used here, with a slight modification. In fact, if we call $\sigma(R)$ a welfare evaluation rule (that is, a relation suitable for welfare analysis), for a relation R, not necessarily complete, x, y two alternatives of the set X. We will show that Strong UPR verifies three axioms, and is the only preference relation that verify it.

Axiom 1: Prudence $x\sigma(R)$ only if xRy.

Axiom 2: Consistency If (x, y) is involved in no cycle of R and xRy, $x\sigma(R)y$.

Axiom 3: Acyclicity $x\sigma(R)y \Rightarrow \nexists(x_i)$ s.th $yRx_1R\ldots Rx_nRx$ with one strict inequality.

Prudence means that the modeler does not use to perform welfare recommendations a preference relation that has not been observed. That is, the modeler cannot discriminate between two incomparable alternatives. Consistency says on the other hand that if a relation is not involved in a cycle and exists, then it should be used to perform welfare analysis. If he knows that a choice has been observed and is preferred, while not being in a cycle, that is, is maximal in a sense, it should be used. Finally acyclicity says that if a preference relation is involved in a cycle, then it should not be used to perform welfare inference, as the modeler does not know in a cycle what is the maximal element.

Theorem 2

A preference relation verifying these three axioms exist and is unique. It is the Strong

UPR.

Proof: It is easy to see that Strong UPR verify these axioms. The problem uniqueness.

Assume two welfare evaluation rule σ, σ' verifying the axioms, then we will show that they coincide.

If the relation R is acyclic, by consistency both relations will coincide.

Now it the relation is cyclic, several cases are possible for two elements x, y:

1. They are incomparable, and by prudence, the welfare evaluation rule won't give any relations between them.
2. (x, y) is not involved in any cycle of R and xRy (wlog), then we have both $x\sigma(R)y$ and $x\sigma'(R)y$ by consistency.
3. (x, y) is involved in a cycle of R and xRy (wlog), then by acyclicity, we have neither $x\sigma(R)y$ nor $x\sigma'(R)y$

These three cases are all the possible case for two elements, so that we have shown that the welfare evaluation rule are identical, and thus are Strong UPR. ■

The previous theorem gives theoretical justification to the use of Strong UPR, with three verifiable axioms.

2.5 Links with other relations of the literature

The two relations introduced here are similar in the reasoning to the *minimum cost index* of Dean and Martin (forthcoming). The main difference here is that they remove relations based on the cost of the violations. It means that when they face a cycle, they do not remove all relations involved in that cycle, but only one. It is of course enough to break the cycle, but to obtain uniqueness, they need a way to select the relation to remove, which is the cost of the violation. It means that their framework is for now only adapted to data with prices.

In a similar need to use prices is the *money pump index* of Echenique, Lee, and Shum (2011). They use the famous possibility of money pumping agents with intransitive preferences to attribute a cost to irrationality. This obviously also relies on the ability to get prices and assign a cost.

It is in general different from relations as the swap index as proposed by Apesteguia and Ballester (forthcoming), because their rely on a complete observed preference order. This measure is a way to evaluate both the inconsistencies and the welfare of agents. The aim is to choose the closest preference order to (empirically) observed choices that is consistent (and thus, verifies the GARP). To assess the closeness of a preference order, they look at the number of swaps of alternatives necessary to get the preference order from the observed choices. The problem here is that the order we observe are incomplete, and thus their method is not applicable and the two preferences relations are in general incomparable.

In short, we have introduced to new relations, UPR and Strong UPR, that are different from what already exist in the literature and have a nice feature: they guarantee acyclicity of the final preference relations. It means that if we have a complete preference relation,

	Average	Std Deviation	Minimum	Maximum
Revealed preferences	46.60	0.37	45.21	47.54
Strict UCR	46.12	0.25	44.83	46.75
UPR	46.10	0.26	44.42	46.75
Strong UPR	45.59	0.67	36.75	46.75
Closed UPR	46.29	0.21	44.96	46.92
Closed Strong UPR	45.66	0.67	36.79	46.83

Table 3.1: Average number of relation per bundle, the theoretical maximum is 47 (each bundle of an agent should compare to all others). Closed indicates that it is the transitive closure.

	Average	Median	Std deviation	Maximum
Strict UCR	14.06	12	8.10	46
UPR	14.52	12	8.82	56
Strong UPR	24.62	19	20.27	144

Table 3.2: Statistical characteristics of the number of relations to remove to rationalize the preferences of an agent. Strict UCR removes slightly less but it leaves some cycles.

we will be able to build a preference order and a utility out of them. They also have theoretical foundations, which despites their apparent similarity, are different:

- UPR's foundations lie in the equivalence with Strict UCR on complete data with choice functions, and in the relative conservativeness compared to the latter in an incomplete data environment.
- Strong UPR's justifications comes from 3 axioms that we have explained above: prudence, consistency and acyclicity. These axioms are quite easily verifiable, but as will be shown in the next section and has already been shown in the first, are in general not verified on consumption data.

3 Experimental results with UPR and Strong UPR

In the previous section, we introduced two new relations. Here we test them on the same data as in the section 1, to compare how much relations will be lost compared to what revealed preferences give us. In exchange we will get acyclic relations, which if complete, allow us to build a utility out of the observation. The acyclicity also means that transitive closure is possible. We perform it on the data to see whether it completes enough our observations or not.

Table 3.1 shows that on average the loss due to acyclicity is small (the difference between Strict UCR and UPR is almost negligible). It comes from the fact that as said theoretically, and verified empirically, the UPR and Strict UCR coincide for acyclic agents of Strict UCR. As more than 90% of households verify acyclicity, both relations are almost always the same. When they differ, UPR is a bit more conservative in order to guarantee acyclicity. Strong UPR is farther away, and the minimum can be very far, even though the average is not. Another way to see it is to have a look at the histogram of the absolute

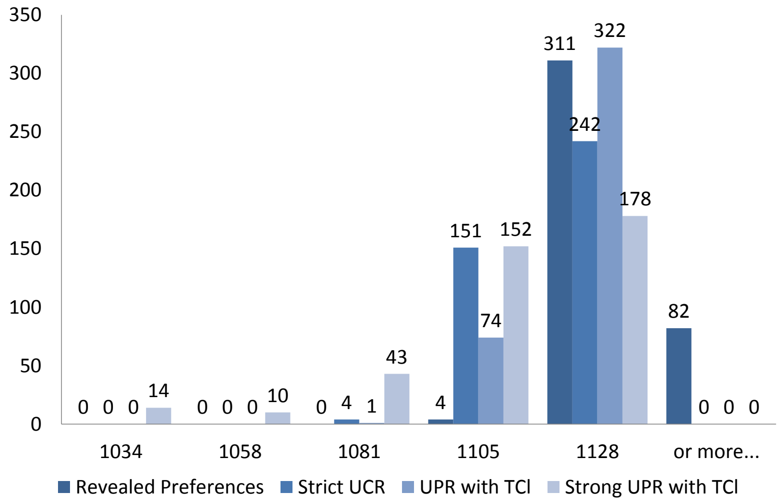

Figure 3.1: Histogram of the absolute number of preference relations per agent. 1128 corresponds to 47 relations per bundle, whereas 1081 corresponds to 46 on average, and so on... Most agents are between 46 and 47 relations per bundle on average. TCl is an abbreviation for transitive closure.

number of relations, as shown in figure 3.1. When a household exhibits more than 1128 preferences relations, it means it does not verify SARP. The histograms also shows us that to rationalize observations, it is not necessary to remove a lot of relations, whereas removing all cycles is more costly, but safer. Closing transitively does not add a lot of relations, so that for readability, only the transitive closure is shown, when meaningful.

Table 3.2 shows the number of relations removed compared to revealed preferences relations. The minimum is not shown because it is always 0 (one agent verify SARP with revealed preferences). The number there are absolute, and they are very small (on average an agent has 1119 preference relations). And they are small despite the number of cycles we might obtain as shown in table 1.1 in 5. It means that a few number of relations generates a lot of cycles, because once cycles are connected together, there are various different ways to obtain new cycles out of them.

UPR is also almost complete, and using the transitive closure allows to get more relations, but does not yield to completeness of the preference relation (the most complete agents misses 2 relations to be complete).

Conclusion

In this work, we tested for the first time Strict UCR on a widely available set of data: consumption data. In our dataset, the preference relation performed quite well, as it resulted in almost complete preference relation, and reduced a lot the number of non rational agents (in the axiom of revealed preference meaning). However some of them remained, due to the incomplete data nature of the experimental data economists have their hands-on. We therefore introduced two new relations, UPR and Strong UPR, that guarantee acyclicity of the preference relation on any kind of dataset. The former is shown to be equivalent or more conservative than Strict UCR, depending on the nature of the data (complete or incomplete), whereas the latter is based on 3 axioms.

These new relations have been tested on data, in section 3, and UPR in particular retain similar results compared to Strict UCR, while being always acyclic. Therefore we are able to build the transitive closure of the latter. Experimentally it means small gains in terms of relations, but does not yield to complete preference relation. In fact, it is almost complete. An open question then is what can be done with this "almost complete" data. Strong UPR is less complete than UPR but has in exchange a very clear stance on how to treat cycles of preferences: we do not infer anything from them. When performing welfare recommendation based on observed choices, Strong UPR restricts to alternatives in between which we have a clear message from the observation. Another way to deal with incompleteness, in particular to get a utility representation, has been explored in Ok (2002).

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4 Appendices

4.1 Appendix 1: Aggregate good

4.2 Demographic description of the sample

Category	Subcategory	Percentage
Age of household head(s)	<35	20%
	$35 \leq age < 65$	67%
	≥ 65	14%
Household composition	1-2 members	40%
	3-4 members	42%
	>4 members	19%
Number of houshold heads	1 head	20%
	2 heads	80%
Household income	$< 20k$	18%
	$20k \leq income < 45k$	82%
	$\geq 45k$	51%
Education of houshold head(s)	No degree	3%
	High school	70%
	College	27%

Table 4.1: Summary of demographic characteristics, percentage are averaged, so that the sum might not be 100%.